Quantum fluctuations of vacuum stress tensors and spacetime curvatures

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Abstract We analyze the quantum fluctuations of vacuum stress tensors and spacetime curvatures, using the framework of linear response theory which connects these fluctuations to dissipation mechanisms arising when stress tensors and spacetime metric are coupled. Vacuum fluctuations of spacetime curvatures are shown to be a sum of two contributions at lowest orders; the first one corresponds to vacuum gravitational waves and is restricted to light-like wavevectors and vanishing Einstein curvature, while the second one arises from gravity of vacuum stress tensors. From these fluctuations, we deduce noise spectra for geodesic deviations registered by probe fields which determine ultimate limits in length or time measurements. In particular, a relation between noise spectra characterizing spacetime fluctuations and the number of massless neutrino fields is obtained.

Keywords Quantum fluctuations; Vacuum stress tensors; Spacetime curvatures; Ultimate sensitivity limits.

INTRODUCTION

Fluctuations of quantum fields lead to observable mechanical effects. Field quanta carry energy and momentum and exert radiation pressure forces upon scatterers [1]. These forces are themselves fluctuating quantities, like stress tensors which describe energy and momentum densities. Such force fluctuations are associated with dissipative forces which damp the motion of scatterers [2].

Those mechanical effects of quantum fields persist in a state with no quanta, i.e. in vacuum. Vacuum fluctuations also exert radiation pressure forces, the so-called Casimir forces, upon scattering boundaries [3]. Such forces fluctuate, like vacuum stress tensors [4]. Relations between fluctuation and dissipation still hold in vacuum, which may be regarded as the zero temperature limit of a thermal equilibrium state [5]. Dissipative motional forces in vacuum may be identified [6] with the effect of radiation of energy into vacuum stemming from non uniform motion of scattering boundaries [7].

When attention is focussed upon questions of principle, it appears that fluctuation-dissipation mechanisms play a fundamental role for determining ultimate limits in quantum measurements. This results for example from an analysis of interferometric length measurements [8] or from a general analysis of the effects of noise and dissipation in high-sensitivity measurements [9]. For measurements performed with endpoints of mass m, radiation pressure exerted by vacuum upon these endpoints imposes [10] a sensitivity limit of the order of Compton wavelength $\frac{\hbar}{mc}$. For macroscopic masses however, i.e. precisely for masses greater than Planck mass $\sqrt{\frac{\hbar c}{G}} \approx 22\mu\text{g}$ where G is Newton constant, Compton wavelength is smaller than Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}}$$

Since it cannot be accepted that sensitivity in spacetime probing goes beyond Planck length (see for example [11]), it appears that gravity has to be taken into account when analysing ultimate limits.

When it is treated in the same spirit as other field theories [12,13], gravitation exhibits quantum metric and curvature fluctuations [14]. Like classical curvature perturbations associated with gravitational waves [15], quantum curvature fluctuations are felt by any field used to probe spacetime. Estimating noise spectra for fluctuating geodesic deviations stemming from these vacuum gravitational waves leads to a universal spectrum for length fluctuations which prevents sensitivity from going beyond Planck length for measurements performed with macroscopic masses [16].

Einstein equation for gravitation [17] can be regarded as a response equation which describes the metric response to a stress tensor perturbation, and used to derive vacuum fluctuations of metric. It also leads to extra curvature fluctuations arising from gravity of quantum fluctuations of vacuum stress tensors [18]. The opinion that gravity only feels mean values of stress tensors has sometimes been expressed [19], but it is known to endanger the consistency of quantum predictions [20]. Even if the existence of quantum fluctuations associated with Einstein equation is denied, unavoidable coupling to vacuum stress tensors lures metric fluctuations into the quantum domain.

Fluctuations of vacuum stress tensors are associated with a dissipative response of vacuum to a metric per-

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turbation that can be identified with particle production in a curved spacetime [21]. It follows that vacuum stress tensors and spacetime metric have to be regarded as dynamical systems coupled through dissipative mechanisms, and that the determination of ultimate quantum limits in spacetime probing requires a consistent treatment of their fluctuations.

In the present paper, we want to give a consistent description of the fluctuations of vacuum stress tensors and spacetime curvatures, of the associated dissipation mechanisms and of the ultimate sensitivity that may be reached in length or time measurements. For that purpose, we shall rely on an analogy with the problem of vacuum radiation pressure acting upon moving scatterers. This analogy has often been used as a guide for studying the interplay between quantum fluctuations and gravitation [22]. Here, we shall apply techniques of linear response theory, which have already proved fruitful for studying moving boundaries coupled to vacuum radiation pressure, to analyse gravitational fluctuations coupled to vacuum stress tensors.

The paper is organized as follows. In a preliminary section, we recall the framework of quantum fluctuations and linear response theory and discuss some questions more specific to gravitational fluctuations. Next sections are devoted to the study of curvature and stress tensor fluctuations. First, we derive the curvature fluctuations associated with Einstein equation for gravitation and recall how fluctuations of vacuum stress tensors may be computed when metric fluctuations are disregarded. General properties of these fluctuations based upon Lorentz invariance and conservation laws in Minkowski spacetime are discussed while explicit results are gathered in appendix A. We then show how to build a consistent description of coupled fluctuations of metric and stress tensors, which is similar to that used for position and force fluctuations for a mirror in vacuum [10] and reflects thermalization of a system coupled to a bath at zero temperature. To this aim, we assume that the low frequency behaviour of gravitation is effectively described by Einstein equation; the possibility that vacuum polarization may modify the effective behaviour of gravitation at low frequency is discussed separately in appendix B. In final sections, we deduce noise spectra for the geodesic deviations registered by a probe field, provide ultimate sensitivity limits that can be attained in length or time measurements, and discuss the obtained results.

QUANTUM FLUCTUATIONS AND LINEAR RESPONSE THEORY

In this preliminary section, we recall general definitions for the correlation and susceptibility functions associated with quantum fluctuations, and briefly discuss some questions which arise when linear response theory is applied to coupled metric and stress tensors.

Correlation functions are defined according to the general prescription

$$\begin{aligned}
C_{AB}(x) &\equiv < A(x)B(0) > - < A(x) >< B(0) > \\
\sigma_{AB}(x) &\equiv \frac{C_{AB}(x) + C_{BA}(-x)}{2\hbar} \\
\xi_{AB}(x) &\equiv \frac{C_{AB}(x) - C_{BA}(-x)}{2\hbar}
\end{aligned}$$

Expectation values are evaluated for free fields in vacuum of quantum field theory in flat spacetime; symmetrised and antisymmetrised functions are vacuum expectation values of anticommutator and commutator respectively. Corresponding spectra are obtained by a translation to momentum domain (same notation will be used throughout the paper)

$$f(x) \equiv \int \frac{\mathrm{d}^d k}{(2\pi)^d} f[k] \exp(-ik_\mu x^\mu)$$

In absence of a more precise specification, spacetime dimension is an arbitrary integer $d \geq 2$. Vacuum is characterized as the equilibrium state at zero temperature, with correlation functions obeying fluctuation-dissipation relations

$$\begin{aligned}
C_{AB}[k] &= 2\hbar\theta(k_0)\sigma_{AB}[k] \\
\xi_{AB}[k] &= \mathrm{sgn}(k_0)\sigma_{AB}[k]
\end{aligned} \qquad \text{(eq1)}$$

where $\theta\,()$ is Heaviside step function, $\mathrm{sgn}\,()$ sign function, and k_0 frequency. These relations mean that vacuum fluctuations do not contain negative frequencies and imply that, for an atom in vacuum, spontaneous transitions correspond to emission but not to absorption. Whilst obvious for vacuum field fluctuations [23], these relations are demonstrated for vacuum stress tensors in a straightforward manner that we briefly recall. We consider, for simplicity, the case of a massless field theory where vacuum fluctuations only contain light-like wavevectors. Stress tensor spectrum is obtained by a convolution product of field spectra: a wavevector k in the stress tensor spectrum is a sum $(k = k' + k'')$ of wavevectors k' and k'' present in a field spectrum. It follows that only positive frequencies appear in noise spectrum $C_{T_{\mu\nu}T_{\rho\sigma}}$ of vacuum stress tensor, and also that this spectrum vanishes when k is a spacelike wavevector [4], since it is built from Lorentz-invariant expressions and contains a factor $\theta(k_0)$. Contributions located on the light cone are allowed, and actually arise in the anomalous case of a two-dimensional spacetime (see appendix A).

Linear response theory has been used to relate dissipative forces experienced by moving boundaries with force fluctuations felt by motionless boundaries [6]. In the case of coupled metric and stress tensors considered here, the same formalism allows one to relate the dissipative response of the stress tensor (or metric tensor) to a perturbation of metric tensor (or stress tensor) with correlation functions computed in the unperturbed case. The

unperturbed case, which is analogous to the motionless case for boundaries, corresponds here to free quantum field theories in flat spacetime. This includes linearized gravitation treated in the spirit of a field theory [12,13], with a metric tensor defined as the sum of the Minkowski tensor $\eta_{\mu\nu} \equiv \mathrm{diag}(1,-1,-1,-1)$ and of a small variation $h_{\mu\nu}$ ($|h_{\mu\nu}| \ll 1$). Throughout the paper, we use the Minkowski tensor for raising and lowering indices as well as for getting traced tensors.

Linear response theory describes responses to a perturbation, which is here characterized as a variation δL of Lagrangian density

$$\delta L(x) = -\frac{1}{2} T_{\mu\nu}(x) h^{\mu\nu}(x) \qquad \text{(eq2)}$$

where $T_{\mu\nu}$ is the stress tensor operator. This may be considered either as a metric perturbation generating a stress tensor response according to the definition of stress tensor in a metric theory of gravitation [24], and representing coupling of metric to non gravitational fields in a linear approximation [13], or as a stress tensor perturbation generating a metric response according to Einstein equation

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad \kappa = \frac{8\pi G}{c^2} \qquad \text{(eq3)}$$

where $G_{\mu\nu}$ is Einstein curvature tensor.

Linear response theory tells us how the susceptibility functions $\chi_{h_{\mu\nu}h_{\rho\sigma}}$ and $\chi_{T_{\mu\nu}T_{\rho\sigma}}$, which describe respectively metric and stress tensor responses, are related to the correlation functions evaluated in flat spacetime. Retarded and advanced susceptibility functions, conveniently written in momentum domain, are sums of dispersive and dissipative parts defined as even and odd parts respectively

$$\begin{aligned} \chi^{\mathrm{ret}}_{AB}[k] &= \overline{\chi}_{AB}[k] + i\xi_{AB}[k] \\ \chi^{\mathrm{adv}}_{AB}[k] &= \chi^{\mathrm{ret}}_{AB}[-k] = \overline{\chi}_{AB}[k] - i\xi_{AB}[k] \end{aligned} \qquad \text{(eq4)}$$

Here A and B stand for components of metric or stress tensors. Dissipative parts ξ_{AB} of these functions are just the commutators yet appearing in the correlation functions. Response function built according to Feynman prescription is the retarded susceptibility for positive frequencies and the advanced one for negative frequencies

$$\begin{aligned} \chi_{AB}[k] &= \theta(k_0)\chi^{\mathrm{ret}}_{AB}[k] + \theta(-k_0)\chi^{\mathrm{adv}}_{AB}[k] \\ &= \overline{\chi}_{AB}[k] + i\sigma_{AB}[k] \end{aligned}$$

We have used the fact that ξ_{AB} is an odd function of k and $\overline{\chi}_{AB}$ an even one, and used relations (1) for expressing χ_{AB} in terms of σ_{AB}. For the problem of coupled metric and stress tensors, dissipative functions σ_{AB} and ξ_{AB} describe particle production in a curved spacetime [21]. The conditions associated with positiveness of dissipation will be exhibited later on.

The dispersive function $\overline{\chi}_{AB}$ may in principle be deduced from the commutator ξ_{AB} through dispersion relations, since causality implies that the retarded susceptibility is analytic and regular when frequency lies in the upper half plane $\mathrm{Im}\, k_0 > 0$. In the case of coupled metric and stress tensors, dispersion relations involve divergences [25] which usually lead to ambiguities in the extraction of a finite part [26]. Dissipative functions are however unambiguously defined and finite [27,28], and we shall focus our attention on fluctuations and dissipation and disregard difficulties associated with dispersion relations.

Unavoidable questions are also those of vacuum stability in presence of gravitation [29] and of renormalisability [30]. Presently, a complete and consistent description of quantum gravity is not available, and its dynamics at high frequencies is poorly understood. In the present paper, we will restrict our interest to fluctuations and dissipation at experimentally accessible frequencies, i.e. at frequencies much lower than Planck scale, and to their effects upon ultimate sensitivity limits. In this restricted context, we shall see that the well-known properties discussed in the present section allow one to obtain significant results, despite of the unsolved problems of quantum gravity. More precisely, we shall only use Einstein equation (3), which describes effective gravitation at low frequencies and is directly connected to the form (2) of the perturbation, and relations (1,4) between response functions and fluctuations. For properly defined physical quantities, computation will reveal that there is no ghost in vacuum.

PROPER FLUCTUATIONS OF CURVATURE

As already stated, Einstein equation can be interpreted as describing metric response to a stress tensor perturbation and thus used to obtain quantum fluctuations of metric tensor. In the present section, we write these metric fluctuations and thereafter deduce curvature fluctuations, which present the advantage over metric fluctuations to be gauge-independent; notice that, for gravity, a gauge transformation is a coordinate transformation.

We first write Einstein curvature tensor $G_{\mu\nu}$ in a linear approximation in $h_{\mu\nu}$ (from now on, quantities like $h_{\mu\nu}$ or $G_{\mu\nu}$ are written in momentum domain)

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \equiv \eta_{\mu\nu\rho\sigma} R^{\rho\sigma} \\ \eta_{\mu\nu\rho\sigma} &\equiv \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}) \\ R_{\mu\nu} &= \frac{1}{2}(k^2 h_{\mu\nu} + k_\mu k_\nu h - k_\mu k^\sigma h_{\nu\sigma} - k_\nu k^\sigma h_{\mu\sigma}) \end{aligned}$$

Here $R_{\mu\nu}$ is linearized Ricci tensor and $h = \eta^{\mu\nu} h_{\mu\nu}$.

The linearized solution of Einstein equation (3) may thus be written in a well chosen gauge, such that $2k^\mu h_{\mu\nu} = k_\nu h$

$$k^2 h^{\mu\nu} = 2R^{\mu\nu} = 2\overline{\eta}^{\mu\nu\rho\sigma}\kappa T_{\rho\sigma}$$
$$\overline{\eta}^{\mu\nu\rho\sigma} \equiv \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{d-2}\eta^{\mu\nu}\eta^{\rho\sigma})$$

This solution may be written as the following response equation

$$h^{\mu\nu} = \chi_{h^{\mu\nu}h^{\rho\sigma}} T_{\rho\sigma}$$

where appears a Feynman response function $\chi_{h^{\mu\nu}h^{\rho\sigma}}$

$$\chi_{h^{\mu\nu}h^{\rho\sigma}} = \frac{2\kappa}{k^2 - i\varepsilon}\overline{\eta}^{\mu\nu\rho\sigma}$$

Using equations (4), this corresponds to a correlation function $\sigma_{h^{\mu\nu}h^{\rho\sigma}}$

$$\sigma_{h^{\mu\nu}h^{\rho\sigma}} = 2\pi\kappa\delta(k^2)\overline{\eta}^{\mu\nu\rho\sigma}$$

With other gauge choices, $\sigma_{h^{\mu\nu}h^{\rho\sigma}}$ may be written as a sum of a transverse part, which is deduced from the preceding expression, and of a gauge-dependent longitudinal part (represented by dots)

$$\sigma_{h^{\mu\nu}h^{\rho\sigma}} = \pi\kappa\delta(k^2)(\pi^{\mu\rho}\pi^{\nu\sigma} + \pi^{\mu\sigma}\pi^{\nu\rho} - \frac{2}{d-2}\pi^{\mu\nu}\pi^{\rho\sigma}) + ...$$

that is also

$$\sigma_{h_{\mu\nu}h_{\rho\sigma}} = 2\pi\kappa\delta(k^2)\sum(\lambda_r \pi^r_{\mu\nu\rho\sigma}) + ...$$
$$\lambda_1 = -\frac{1}{d-2} \qquad \lambda_0 = 1 \qquad \text{(eq5)}$$

In these expressions, $\pi_{\mu\nu}$ is the tensorial form of a Lorentz-invariant transverse tensor

$$\pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

and $\pi^r_{\mu\nu\rho\sigma}$ ($r = 0, 1$) are the two orthogonal tensorial forms taken by correlation functions of such a tensor

$$\pi^r_{\mu\nu\rho\sigma} = \alpha_r \pi_{\mu\nu}\pi_{\rho\sigma} + \beta_r(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho}) \qquad \text{(eq6)}$$
$$\alpha_1 = \frac{1}{d-1} \qquad \beta_1 = 0$$
$$\alpha_0 = -\frac{1}{d-1} \qquad \beta_0 = \frac{1}{2}$$

More details about these two forms are given in the next section.

We now evaluate curvature fluctuations. Longitudinal terms do not contribute to curvature and transverse fluctuations are concentrated upon the light cone, as for any massless field theory. One checks that, as expected, fluctuations of Einstein tensor vanish. Curvature fluctuations may therefore be considered as vacuum gravitational waves [14], characterized like classical gravitational waves by a non vanishing Riemann curvature and a vanishing Ricci curvature.

Using the expression of linearized Riemann curvature

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(k_\mu k_\rho h_{\nu\sigma} + k_\nu k_\sigma h_{\mu\rho} - k_\nu k_\rho h_{\mu\sigma} - k_\mu k_\sigma h_{\nu\rho})$$
$$= \frac{1}{2}(k_\mu \eta_{\nu\lambda} - k_\nu \eta_{\mu\lambda})(k_\rho \eta_{\sigma\tau} - k_\sigma \eta_{\rho\tau})h^{\lambda\tau}$$

one gets from equations (5)

$$\sigma_{R_{\mu\nu\rho\sigma}R_{\mu'\nu'\rho'\sigma'}} = a\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}_{\mu'\nu'\rho'\sigma'} + b(\mathcal{R}_{\mu\nu\mu'\nu'}\mathcal{R}_{\rho\sigma\rho'\sigma'} + \mathcal{R}_{\mu\nu\rho'\sigma'}\mathcal{R}_{\rho\sigma\mu'\nu'}) \qquad \text{(eq7)}$$

Coefficients $\mathcal{R}_{\mu\nu\rho\sigma}$ correspond to Riemann curvatures evaluated for a conformal metric ($h_{\mu\nu} = \eta_{\mu\nu}$)

$$\mathcal{R}_{\mu\nu\rho\sigma} = \frac{1}{2}(k_\mu \eta_{\nu\lambda} - k_\nu \eta_{\mu\lambda})(k_\rho \eta_{\sigma\tau} - k_\sigma \eta_{\rho\tau})\eta^{\lambda\tau}$$
$$= \frac{1}{2}(k_\mu k_\rho \eta_{\nu\sigma} + k_\nu k_\sigma \eta_{\mu\rho} - k_\nu k_\rho \eta_{\mu\sigma} - k_\mu k_\sigma \eta_{\nu\rho})$$

Coefficients a and b are

$$a = 2\pi\kappa\delta(k^2)\sum(\lambda_r\alpha_r) = -\frac{2}{d-2}\pi\kappa\delta(k^2)$$
$$b = 2\pi\kappa\delta(k^2)\sum(\lambda_r\beta_r) = \pi\kappa\delta(k^2) \qquad \text{(eq8)}$$

One may consider these equations as a Lorentz-invariant and gauge-independent form of already known expressions for vacuum gravitational waves (see for example metric fluctuations evaluated in transverse traceless gauge in [14]).

FLUCTUATIONS AND DISSIPATIVE POLARIZATION OF VACUUM STRESS TENSORS

We now study the correlation functions characterizing quantum fluctuations of vacuum stress tensors of non gravitational fields. We then express the dissipative response of vacuum stress tensor to a metric perturbation in terms of these correlation functions.

The correlation functions of vacuum stress tensors are evaluated in Minkowski spacetime according to the general prescription of linear response theory (see above). Their tensorial properties follow from symmetries of stress tensor, Lorentz invariance and conservation laws for vacuum fields in Minkowski spacetime. As a consequence of the latter property $T_{\mu\nu}$ is divergenceless, i.e. transverse in momentum domain, and correlation functions obey

$$k^\mu \sigma_{T_{\mu\nu}T_{\rho\sigma}} = k^\rho \sigma_{T_{\mu\nu}T_{\rho\sigma}} = 0 \qquad \text{(eq9)}$$

It follows [21] that spectra decompose over the two transverse tensorial forms π_r defined by equations (6)

$$\sigma_{T_{\mu\nu}T_{\rho\sigma}} = (k^2)^2\sigma\sum(\zeta_r \pi^r_{\mu\nu\rho\sigma})$$
$$\sigma = \hbar\pi(k^2)^{d/2-2}\theta(k^2) \qquad \text{(eq10)}$$

Note however that an exception arises for $d = 2$. Extra terms are indeed allowed for wavevectors localized on the light cone, and such terms effectively contribute for scalar fields in two-dimensional spacetime (see appendix A).

The two tensorial forms π^r are orthogonal projectors onto the subspace of transverse tensors. They obey simple rules for product and trace operations

$$(\pi^r \cdot \pi^s)_{\mu\nu\rho\sigma} \equiv \pi^{r\ \lambda\tau}_{\mu\nu} \pi^s_{\lambda\tau\rho\sigma} = \delta_{rs} \pi^r_{\mu\nu\rho\sigma}$$
$$(\pi^1)^{\mu}_{\ \mu\rho\sigma} = \pi_{\rho\sigma} \qquad (\pi^0)^{\mu}_{\ \mu\rho\sigma} = 0$$

Applying these two projectors onto the stress tensor, one therefore gets a decomposition of its fluctuations as a sum of two uncorrelated components

$$\begin{aligned} T_{\mu\nu} &= T^1_{\mu\nu} + T^0_{\mu\nu} \\ T^1_{\mu\nu} &= (\pi^1 \cdot T)_{\mu\nu} = \pi^{1\ \rho\sigma}_{\mu\nu} T_{\rho\sigma} = \frac{1}{d-1} \pi_{\mu\nu} T \\ T^0_{\mu\nu} &= (\pi^0 \cdot T)_{\mu\nu} = \pi^{0\ \rho\sigma}_{\mu\nu} T_{\rho\sigma} = T_{\mu\nu} - \frac{1}{d-1} \pi_{\mu\nu} T \end{aligned}$$

Component $T^1_{\mu\nu}$ is proportional to the traced tensor ($T \equiv \eta^{\rho\sigma} T_{\rho\sigma}$) while component $T^0_{\mu\nu}$ is traceless. Contributions to noise of these two components correspond to the two parts ($r = 0, 1$) in equation (10).

The two coefficients ζ_r ($r = 0, 1$) appearing in equations (10) are dimensionless functions of k^2, and pure numbers for massless field theories. Note that we use natural spacetime units with $c = 1$, but we keep $\hbar$ as a scale for quantum fluctuations in equations (10) and similar forthcoming expressions. Explicit expressions of these coefficients computed for scalar fields and Maxwell fields in a spacetime of arbitrary dimension are given in appendix A. In the following, we shall consider ζ_r as the sum of contributions of all non gravitational fields, restricting our interest to massless fields which correspond to a long range polarization. We will touch on gravitational contribution later on.

Now, the dissipative part of the stress tensor response to a metric perturbation is described by

$$< \delta T_{\mu\nu} >_{\mathrm{dissip}} = \sigma_{T_{\mu\nu} T_{\rho\sigma}} h^{\rho\sigma} \equiv (\sigma_{TT} \cdot h)_{\mu\nu}$$

This is the dissipative part of Feynman response function; for retarded and advanced responses, σ_{TT} has to be replaced by ξ_{TT} (for the sake of readibility, we occasionally omit indices). As a consequence of relations (9), this expression vanishes for metric variations corresponding to coordinate transformations

$$h^{\rho\sigma} = -i(k^\rho \xi^\sigma + k^\sigma \xi^\rho) \quad \rightarrow \quad < \delta T_{\mu\nu} >_{\mathrm{dissip}} = 0$$

In other words, dissipative part of vacuum stress tensor polarization has an intrinsically geometrical (i.e. gauge-independent) character, which appears closely connected to energy conservation for vacuum fluctuations in Minkowski spacetime. It is not affected by coordinate transformations and may thus be written in terms of curvatures only.

Using equation (10), we may obtain explicit expressions of vacuum stress tensor polarization in terms of spacetime curvatures, more precisely of the two independent transverse tensors $G^r_{\mu\nu}$ that can be built from Einstein tensor ($G^r \equiv \pi^r \cdot G$; λ_r defined in equations 5)

$$< \delta T_{\mu\nu} >_{\mathrm{dissip}} = 2k^2 \sigma \sum (\lambda_r \zeta_r G^r_{\mu\nu}) \qquad \text{(eq11)}$$

Component $G^1_{\mu\nu}$ is proportional to traced Einstein curvature ($G = \eta^{\mu\nu} G_{\mu\nu}$), or equivalently to scalar curvature ($R = \eta^{\mu\nu} R_{\mu\nu}$)

$$G^1_{\mu\nu} = \frac{1}{d-1} \pi_{\mu\nu} G \qquad G = -\frac{d-2}{2} R$$

In contrast, $G^0_{\mu\nu}$ is traceless and can be written in terms of Weyl curvature

$$G^0_{\mu\nu} = G_{\mu\nu} - \frac{1}{d-1} \pi_{\mu\nu} G = \frac{d-2}{d-3} \frac{k^\rho k^\sigma}{k^2} W_{\mu\rho\nu\sigma}$$

Weyl tensor vanishes for $d = 2$ or $d = 3$ and is otherwise defined by

$$\begin{aligned} W_{\mu\rho\nu\sigma} &= R_{\mu\rho\nu\sigma} \\ &\quad - \frac{1}{d-2} (\eta_{\rho\sigma} R_{\mu\nu} + \eta_{\mu\nu} R_{\rho\sigma} - \eta_{\rho\nu} R_{\mu\sigma} - \eta_{\mu\sigma} R_{\rho\nu}) \\ &\quad + \frac{1}{d-1} \frac{1}{d-2} (\eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\mu\sigma} \eta_{\rho\nu}) R \end{aligned}$$

These two components thus have different behaviours with respect to conformal metric perturbations ($h_{\mu\nu} \sim \eta_{\mu\nu}$), since only $G^1_{\mu\nu}$ differs from zero in this case. Note also that only $G^0_{\mu\nu}$ contributes to stress tensor polarization for conformally invariant field theories (see appendix A).

It may be emphasized that vacuum stress tensor polarization (11) cannot be written as a local expression [28], because of factor $\theta(k^2)$ appearing in expression (10) of σ. Using fluctuation-dissipation relations (1) and translating from momentum domain to spacetime domain, one shows that it has rather a causally propagating form with retarded and advanced responses propagating on the light cone. In fact, this factor $\theta(k^2)$ may be considered as expressing causality of vacuum response to a metric perturbation.

As discussed in the introduction, dissipative vacuum stress tensor polarization may be identified with particle production in response to a gravitational perturbation. It is thus expected to obey some positiveness conditions which imply that perturbation is damped by back-reaction of particle production. In order to exhibit such conditions, it is worth studying energy-momentum transfer between matter fields and gravity [21]. As well known, one can define a pseudo stress tensor, say $\Theta_{\mu\nu}$, for gravitational field, which is such that the total stress tensor $T_{\mu\nu} + \Theta_{\mu\nu}$ obeys an ordinary energy-momentum conservation law [31]. Since the matter stress tensor $T_{\mu\nu}$

has a null covariant divergence, one obtains the energy-momentum transfer as

$$\partial^\nu T_{\mu\nu}(x) = -\partial^\nu \Theta_{\mu\nu}(x)$$
$$= \Gamma_\nu^{\lambda\nu}(x) T_{\mu\lambda}(x) + \Gamma_\mu^{\lambda\nu}(x) T_{\nu\lambda}(x)$$

where $\Gamma_\mu^{\lambda\nu}$ $(= \eta^{\nu\rho}\Gamma_{\mu\rho}^\lambda)$ are Christoffel symbols. This transfer vanishes at first order in the metric tensor, because the linearized stress tensor has a null divergence and $\Theta_{\mu\nu}$ is a quadratic expression of the metric tensor. It may be evaluated at second order by replacing on the right-hand side the Christoffel symbols and stress tensors by their linearized expressions. One thus obtains the energy-momentum transfer Π_μ integrated over spacetime as

$$\Pi_\mu = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \hbar k_\mu \,\mathrm{sgn}(k_0) n[k]$$
$$n[k] = \frac{1}{2\hbar} \sigma_{T_{\lambda\nu} T_{\rho\sigma}}[k] h^{\rho\sigma}[k] h^{\lambda\nu}[-k] \tag{eq12}$$

This integrated transfer Π_μ characterizes energy-momentum dissipation due to particle production in response to a metric perturbation; $\hbar k_\mu$ is the energy-momentum transfer per produced particle while $n[k]$ is the density of particles produced at a given wavevector. Due to transversality (9) of stress tensor correlation functions, function $n[k]$ may be written as a scalar quadratic form of curvatures

$$n[k] = 2\pi (k^2)^{d/2-2} \theta(k^2) \sum (\zeta_r \lambda_r^2 G^{r\ \mu\nu}[k] G^r_{\mu\nu}[-k])$$

with

$$4G^{1\ \mu\nu} G^1_{\mu\nu} = \frac{(d-2)^2}{d-1} R^2$$
$$4G^{0\ \mu\nu} G^0_{\mu\nu} = \frac{d-2}{d-3} W^{\mu\rho\nu\sigma} W_{\mu\rho\nu\sigma}$$
$$= R^{\mu\rho\nu\sigma} R_{\mu\rho\nu\sigma} - \frac{R^2}{d-1}$$

Positiveness of dissipated energy-momentum thus appears to be associated with positiveness of the coefficients ζ_r. One effectively checks that coefficients ζ_r given in appendix A are positive.

QUANTUM FLUCTUATIONS OF COUPLED METRIC AND STRESS TENSORS

Stress tensor and spacetime curvature are coupled dynamical systems. Up to now, we have studied the proper fluctuations of each system, as well as the response of a system to a perturbation due to the other one. The purpose of the present section is to give a consistent description at lowest orders of coupled fluctuations and of interlinked response mechanisms. Stress tensor responds to curvatures, i.e. to transverse components of the metric tensor. We may therefore restrict the discussion to transverse parts of the correlation functions and deal with the two orthogonal components $(r = 0, 1)$ separately.

We first write coupled stress tensor fluctuations as sums of corresponding input fluctuations $(T_{\mu\nu}^{r\ \mathrm{in}})$ and of linear response to metric fluctuations

$$T^r_{\mu\nu} = T^{r\ \mathrm{in}}_{\mu\nu} + \chi^{r\ \mathrm{in}}_{TT} h^r_{\mu\nu} \tag{13a}$$

Input fluctuations are characterized by correlation functions (10) where the subscript "in" did not appear

$$\sigma_{T_{\mu\nu}T_{\rho\sigma}}{}^{\mathrm{in}} = \sum (\sigma^{r\ \mathrm{in}}_{TT} \pi^r_{\mu\nu\rho\sigma}) \qquad \sigma^{r\ \mathrm{in}}_{TT} = (k^2)^2 \zeta_r \sigma$$

Linear susceptibility $\chi^{r\ \mathrm{in}}_{TT}$ is the lowest order vacuum stress tensor polarization

$$\chi^{r\ \mathrm{in}}_{TT} = (k^2)^2 \zeta_r \Gamma_r = (k^2)^2 \zeta_r (\overline{\Gamma}_r + i\sigma)$$

Correlation function $\sigma^{r\ \mathrm{in}}_{TT}$ is the imaginary part of the linear susceptibility $\chi^{r\ \mathrm{in}}_{TT}$. As already discussed, we will not make use of the detailed form of the dispersive part $\overline{\Gamma}_r$.

In a similar manner, we write the linear response of metric to stress tensor fluctuations

$$h^r_{\mu\nu} = h^{r\ \mathrm{in}}_{\mu\nu} + \chi^{r\ \mathrm{in}}_{hh} T^r_{\mu\nu} \tag{13b}$$

Input fluctuations $h^{r\ \mathrm{in}}_{\mu\nu}$ are characterized by a correlation function $\sigma^{r\ \mathrm{in}}_{hh}$ which is the imaginary part of the linear susceptibility $\chi^{r\ \mathrm{in}}_{hh}$ (longitudinal terms are omitted)

$$\sigma_{h_{\mu\nu}h_{\rho\sigma}}{}^{\mathrm{in}} = \sum (\sigma^{r\ \mathrm{in}}_{hh} \pi^r_{\mu\nu\rho\sigma}) \qquad \sigma^{r\ \mathrm{in}}_{hh} = 2\pi\kappa\lambda_r \delta(k^2)$$
$$\chi^{r\ \mathrm{in}}_{hh} = \frac{2\kappa\lambda_r}{k^2 - i\varepsilon} = \frac{2\kappa\lambda_r}{k^2} - 1 + i\sigma^{r\ \mathrm{in}}_{hh}$$

One easily solves equations (13) to obtain coupled fluctuations in terms of input ones

$$T^r_{\mu\nu} = \chi^r_{Th} T^{r\ \mathrm{in}}_{\mu\nu} + \chi^r_{TT} h^{r\ \mathrm{in}}_{\mu\nu}$$
$$h^r_{\mu\nu} = \chi^r_{Th} h^{r\ \mathrm{in}}_{\mu\nu} + \chi^r_{hh} T^{r\ \mathrm{in}}_{\mu\nu}$$

with

$$\chi^r_{hh} = \frac{1}{(\chi^{r\ \mathrm{in}}_{hh})^{-1} - \chi^{r\ \mathrm{in}}_{TT}}$$
$$\chi^r_{Th} = \frac{1}{1 - \chi^{r\ \mathrm{in}}_{TT} \chi^{r\ \mathrm{in}}_{hh}}$$
$$\chi^r_{TT} = \frac{1}{(\chi^{r\ \mathrm{in}}_{TT})^{-1} - \chi^{r\ \mathrm{in}}_{hh}}$$

One deduces the correlation functions for coupled fluctuations

$$\sigma_{h_{\mu\nu}h_{\rho\sigma}} = \sum (\sigma^r_{hh} \pi^r_{\mu\nu\rho\sigma})$$
$$\sigma^r_{hh} = \chi^r_{hh} \sigma^{r\ \mathrm{in}}_{TT} (\chi^r_{hh})^* + \chi^r_{Th} \sigma^{r\ \mathrm{in}}_{hh} (\chi^r_{Th})^*$$

Fluctuation-dissipation relations for coupled variables follow from those known for input ones: coupled correlation function σ^r_{hh} is the imaginary part of coupled susceptibility χ^r_{hh}, and coupled correlation functions obey relations (1) characteristic of vacuum. Similar results are obtained for stress tensor correlation functions and cross correlations between stress tensor and metric.

Writing coupled susceptibilities as

$$\chi^r_{hh} = \gamma_r \chi^{r\ \mathrm{in}}_{hh} \qquad \chi^r_{TT} = \gamma_r \chi^{r\ \mathrm{in}}_{TT}$$
$$\chi^r_{Th} = \gamma_r = \frac{1}{1 - 2\kappa\lambda_r\zeta_r k^2 \Gamma_r}$$

one sees that $\kappa\gamma_r$ is a momentum-dependent effective gravitational constant for the $r-$component, which may be compared with the effective mass for an unbound mirror coupled to vacuum radiation pressure [10]. Equal low frequency values $\gamma_1[0]$ and $\gamma_0[0]$, differing from the standard value 1, could be dealt with by redefining κ. A difference between $\gamma_1[0]$ and $\gamma_0[0]$ would lead to an effective gravitation differing from the predictions of general relativity. This possibility is discussed in appendix B. Since accurate experiments have checked that gravitation is consistent with Einstein theory [32], we will assume in the following that

$$\gamma_1[0] = \gamma_0[0] = 1$$

We remark that coupled metric fluctuations may be written

$$\sigma^r_{hh} = \sigma^{r\ \mathrm{in}}_{hh} + \left(\frac{2\kappa\lambda_r}{k^2}\right)^2 \sigma^r_{TT}$$

The former term coincides with proper metric fluctuations while the latter appears to result from gravity of vacuum stress tensor. As expected, these two contributions have been included in a consistent treatment. At this point, it is worth recalling that metric fluctuations have a non-commutative character as vacuum stress tensor fluctuations (see equations 1), and noting that metric and stress tensor fluctuations are correlated in the coupled system.

We now restrict the discussion to momenta much lower than Planck frequency, where correlation functions have simple approximated forms

$$\sigma_{h_{\mu\nu}h_{\rho\sigma}} = \sigma_{h_{\mu\nu}h_{\rho\sigma}}{}^{\mathrm{in}} + 4\kappa^2\sigma \sum (\lambda_r^2 \zeta_r \pi^r_{\mu\nu\rho\sigma})$$
$$\sigma_{T_{\mu\nu}T_{\rho\sigma}} = \sigma_{T_{\mu\nu}T_{\rho\sigma}}{}^{\mathrm{in}} = (k^2)^2 \sigma \sum (\zeta_r \pi^r_{\mu\nu\rho\sigma})$$
$$\sigma_{T_{\mu\nu}h_{\rho\sigma}} = 2\kappa k^2 \sigma \sum (\lambda_r \zeta_r \pi^r_{\mu\nu\rho\sigma})$$

We deduce correlations of Einstein curvature and stress tensor

$$\sigma_{G_{\mu\nu}G_{\rho\sigma}} = \kappa^2 \sigma_{T_{\mu\nu}T_{\rho\sigma}}$$
$$\sigma_{G_{\mu\nu}T_{\rho\sigma}} = \kappa \sigma_{T_{\mu\nu}T_{\rho\sigma}}$$

which can be summarized by simple identities for stochastic variables

$$G_{\mu\nu} = \kappa T_{\mu\nu} = \kappa T_{\mu\nu}{}^{\mathrm{in}}$$

We eventually obtain correlation functions for Riemann curvature fluctuations. Their form is still given by equations (7) with coefficients a and b being sums of proper terms (8) and of terms arising from gravity of vacuum stress tensor

$$a = -\frac{2}{d-2}\pi\kappa\delta(k^2) + 4\kappa^2\sigma \sum (\lambda_r^2 \zeta_r \alpha_r)$$
$$b = \pi\kappa\delta(k^2) + 4\kappa^2\sigma \sum (\lambda_r^2 \zeta_r \beta_r) \qquad (14)$$

In these expressions, ζ_r is a sum of contributions of non gravitational fields, which we may restrict for simplicity to massless fields. The gravitational contribution deserves a specific treatment. The pseudo stress tensor $\Theta_{\mu\nu}$ for gravitational field is a quadratic expression of the metric tensor, but it has the same magnitude as the stress tensor $T_{\mu\nu}$ associated with non gravitational fields, and must in principle be taken into account in the analysis of vacuum stress tensor fluctuations and associated polarization. At first sight, it seems natural to conclude that coefficients ζ_r have to be modified in order to include a gravitational contribution [33]. The non linear nature of gravitation however entails a second correction to the previously computed expressions. Without entering into a detailed discussion of these corrections, we want to emphasize the following point. Since $\Theta_{\mu\nu}$ is directly related to the non linear correction to Einstein tensor, the already written expressions are a correct description of the fluctuations of Einstein and Ricci curvatures in terms of non gravitational stress tensors only.

QUANTUM LIMITS IN SPACETIME PROBING

In this final section, we analyse in detail how curvature fluctuations result in sensitivity limits in spacetime probing.

For that purpose, we consider the specific case of four-dimensional spacetime and rewrite Riemann curvature fluctuations

$$C_{R_{\mu\nu\rho\sigma}R_{\mu'\nu'\rho'\sigma'}} = 4\beta(\mathcal{R}_{\mu\nu\mu'\nu'}\mathcal{R}_{\rho\sigma\rho'\sigma'} + \mathcal{R}_{\mu\nu\rho'\sigma'}\mathcal{R}_{\rho\sigma\mu'\nu'})$$
$$- 4\alpha\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}_{\mu'\nu'\rho'\sigma'}$$

where functions α and β are obtained by using equations (1), (7), (14) and definitions (5), (6) and (10), and by considering only massless fields, i.e. the Maxwell field and N_ν massless neutrino fields which each contribute for one fourth of the contribution of Maxwell field [28]

$$\alpha = 4\pi^2 l_P^2 \theta(k_0) \left(\delta(k^2) + \frac{4+N_\nu}{30\pi} l_P^2 \theta(k^2)\right)$$
$$\beta\ = 4\pi^2 l_P^2 \theta(k_0) \left(\delta(k^2) + \frac{4+N_\nu}{20\pi} l_P^2 \theta(k^2)\right)$$

More precisely, N_ν has to be understood as the number of neutrino fields with a mass smaller than noise frequencies (see appendix A); note also that ζ_1 vanishes for massless fields, as a consequence of conformal invariance. Noise spectra $C_{R_{\mu\nu\rho\sigma}R_{\mu'\nu'\rho'\sigma'}}$ contain components proportional to $\delta(k^2)$ which correspond to vacuum gravitational waves and scale as $(kl_P)^2$, and components proportional to $\theta(k^2)$ which arise from gravity of vacuum stress tensors and scale as $(kl_P)^4$. The latter have a smaller magnitude at low momenta, but are present in a larger momentum domain.

In length or time measurements using a probe field, for example in interferometric or timing measurements, the field phase registers metric perturbations along propagation. Using the law of geodesic deviation [15], this effect is described by a deviation tensor $\Delta_{\mu\rho}$ in the eikonal approximation and at first order in curvature

$$\begin{aligned}\Delta_{\mu\rho} &\equiv \frac{1}{K_0}\frac{\partial K_\rho}{\partial x^\mu} \equiv \frac{1}{K_0}\frac{\partial K_\mu}{\partial x^\rho} \\ &= \int_0^\tau Q_{\mu\rho}\left(x - t\frac{K}{K_0}\right)\mathrm{d}t \\ Q_{\mu\rho} &= R_{\mu\nu\rho\sigma}\frac{K^\nu K^\sigma}{K_0^2}\end{aligned}$$

The integral is evaluated along a one-way track (other measurement techniques are discussed in [16,34] and references therein), the coordinate time t is used as an affine parameter and τ is the time of propagation from the emitter to the receiver. K_ρ is the wavevector of the probe field and K_0 its frequency; K_ρ is related to the gradient of the probe phase, or equivalently to the four-dimensional velocity vector for the probe beam. Notice that the deviation tensor $\Delta_{\mu\rho}$ and the tidal tensor $Q_{\mu\rho}$ are homogeneous functions of K, in the eikonal approximation. The effect of curvature fluctuations upon length or time measurements may then be characterized by noise spectra for components of the deviation tensor

$$\begin{aligned}C_{\Delta_{\mu\nu}\Delta_{\rho\sigma}}[\omega] &= \left(\frac{\tau}{2\pi}\right)^2\int\frac{\mathrm{d}\mathbf{k}^2\mathrm{d}k_3}{2} \\ &\times < C_{Q_{\mu\nu}Q_{\rho\sigma}} >_{\omega,\mathbf{k}^2,k_3}\operatorname{sinc}^2\frac{(\omega - k_3 v)\tau}{2}\end{aligned}$$

Deviation tensors are evaluated at a given spatial position. For simplicity, we consider from now on that the probe propagates along the x_3-axis with a normalized velocity v ($K_1 = K_2 = 0$; $K_3 = K_0 v$; $v = 1$ for a massless probe field; $v < 1$ for a massive probe field) and identify the various components as temporal (index 0), longitudinal (index 3) and transverse (index 1 or 2). In the foregoing equation, $\operatorname{sinc}(x)$ stands for $\frac{\sin(x)}{x}$ and $< C_{Q_{\mu\nu}Q_{\rho\sigma}} >_{\omega,\mathbf{k}^2,k_3}$ represents the average of $C_{Q_{\mu\nu}Q_{\rho\sigma}}[k]$ over azimut angle with the constraints $k_0 = \omega$ and $k_1^2 + k_2^2 + k_3^2 = \mathbf{k}^2$.

Tidal tensor $Q_{\mu\rho}$ is obtained through a contraction of the Riemann tensor defined with respect to the propagation direction of the probe. Temporal components of the deviation tensor may be expressed in terms of longitudinal ones

$$\begin{aligned}Q_{03} &= vQ_{33} \qquad Q_{00} = v^2 Q_{33} \\ Q_{01} &= vQ_{13} \qquad Q_{02} = vQ_{23}\end{aligned}$$

As a consequence, temporal components vanish at the limit of slow test particles, where the effect of curvature is reduced to a purely spatial tidal effect. More generally for any velocity, it will be sufficient to study the 6 spatial components of the tidal tensor, whose noise spectra are given by

$$\begin{aligned}\sigma_{Q_{ij}Q_{kl}} &= 4\beta(\mathcal{Q}_{ik}\mathcal{Q}_{jl} + \mathcal{Q}_{il}\mathcal{Q}_{jk}) - 4\alpha\mathcal{Q}_{ij}\mathcal{Q}_{kl} \\ \mathcal{Q}_{ik} &= \frac{1}{2}\left(k_i k_k(1-v^2) - \delta_{ik}(k_0 - k_3 v)^2\right. \\ &\quad \left. - v(\delta_{i3}k_k + k_i\delta_{k3})(k_0 - k_3 v)\right)\end{aligned}$$

Latin indices represent spatial components and δ_{ik} is the Kronecker symbol for such indices. In place of Q_{11} and Q_{22}, we introduce the two variables

$$\begin{aligned}Q'_{12} &= \frac{Q_{11} - Q_{22}}{2} \\ Q = Q^\mu_\mu &= -(Q_{11} + Q_{22}) - Q_{33}(1 - v^2) \\ &= R_{\nu\sigma}\frac{K^\nu K^\sigma}{K_0^2}\end{aligned}$$

Straightforward computations lead to noise spectra characterizing fluctuations of the components of the tidal tensor averaged over azimut angle. It turns out that the 4 components Q_{13}, Q_{23}, Q_{12} and Q'_{12} are uncorrelated stochastic variables with noise spectra given by

$$\begin{aligned}< C_{Q_{13}Q_{13}} > &= < C_{Q_{23}Q_{23}} > \\ &= \frac{\beta - \alpha}{2}(\mathbf{k}^2 - k_3^2)(\omega - k_3 v)^2 \\ &- \frac{2\beta - \alpha}{2}(\mathbf{k}^2 - k_3^2)(\omega^2 - k_3^2)(1 - v^2) \\ &+ \beta(\omega^2 - k_3^2)(\omega - k_3 v)^2 \\ < C_{Q_{12}Q_{12}} > &= < C_{Q'_{12}Q'_{12}} > \\ &= \frac{2\beta - \alpha}{8}(\mathbf{k}^2 - k_3^2)^2(1 - v^2)^2 \\ &+ \beta(\omega - k_3 v)^4 \\ &- \beta(\omega - k_3 v)^2(\mathbf{k}^2 - k_3^2)(1 - v^2)\end{aligned}$$

The 2 remaining components are correlated variables with noise spectra given by

$$\begin{aligned}< C_{Q_{33}Q_{33}} > &= (2\beta - \alpha)(\omega^2 - k_3^2)^2 \\ < C_{Q_{33}Q} > &= -(2\beta - \alpha)(\omega^2 - \mathbf{k}^2)(\omega^2 - k_3^2)(1 - v^2) \\ &- 2(\beta - \alpha)(\mathbf{k}^2 - k_3^2)(\omega - k_3 v)^2 \\ &+ 2\alpha(\omega^2 - \mathbf{k}^2)(\omega - k_3 v)^2 \\ < C_{QQ} > &= (2\beta - \alpha)(\omega^2 - \mathbf{k}^2)^2(1 - v^2)^2 \\ &+ 4(\beta - \alpha)(\omega - k_3 v)^4 \\ &- 4\alpha(\omega^2 - \mathbf{k}^2)(1 - v^2)(\omega - k_3 v)^2\end{aligned}$$

DISCUSSION

Propagation of a massless field is resonantly affected by fluctuations with light-like wavevectors, so that the smaller fluctuations with time-like wavevectors may be disregarded in this case. For a massive probe field in contrast, propagation is not resonantly affected by curvature fluctuations.

If we restrict our attention to the contribution of vacuum gravitational waves (light-like wavevectors), the foregoing expressions are simplified due to the relations $\beta = \alpha$; $\omega^2 = \mathbf{k}^2$. Component Q thus vanishes

$$< C_{Q_{33}Q} >=< C_{QQ} >= 0$$

since Ricci curvature vanishes for gravitational waves. There remain 5 uncorrelated components with spectra given by

$$\begin{aligned}
< C_{Q_{33}Q_{33}} > &= \alpha(\omega^2 - k_3^2)^2 \\
< C_{Q_{13}Q_{13}} > &= < C_{Q_{23}Q_{23}} > \\
&= \alpha(\omega^2 - k_3^2)(\omega - k_3 v)^2 \\
&\quad - \frac{\alpha}{2}(\omega^2 - k_3^2)^2(1 - v^2) \\
< C_{Q_{12}Q_{12}} > &= < C_{Q'_{12}Q'_{12}} > \\
&= \alpha(\omega - k_3 v)^4 \\
&\quad - \alpha(\omega^2 - k_3^2)(1 - v^2)(\omega - k_3 v)^2 \\
&\quad + \frac{\alpha}{8}(\omega^2 - k_3^2)^2(1 - v^2)^2 \\
\alpha &= 4\pi^2 l_P^2 \theta(\omega)\delta(\omega^2 - \mathbf{k}^2)
\end{aligned}$$

Component Q_{33} describes a longitudinal deviation for longitudinally shifted geodesics, which plays a role in interferometric measurements. This is why its noise spectrum has already been studied in the particular case of massless probes ($v = 1$), as a way to the detection of a stochastic background of gravitational waves [34], and in the context of ultimate quantum limits in optical interferometry or in electromagnetic timing experiments [16]. Other components of the tidal and deviation tensors have apparently not been studied up to now. Components Q_{13} and Q_{23} correspond to transverse deviations for longitudinally shifted geodesics, or equivalently to longitudinal deviations for transversely shifted geodesics, that is to a bending of the probe beam. Components Q_{12} and Q'_{12} correspond to transverse deviations for transversely shifted geodesics, that is to a focusing or defocusing effect of the probe beam.

We may emphasize that the foregoing expressions also contain informations about ultimate quantum limits in measurements based upon atomic interferometry ($v < 1$). In the limit of slow probe particles ($v \to 0$) in particular, the geodesic deviation tensor takes a simple form, and the leading contributions to noise spectra due to vacuum gravitational waves may be written

$$C_{\Delta_{33}\Delta_{33}}[\omega] = \frac{32}{15}\omega^3 l_P^2 \sin^2\frac{\omega\tau}{2}\theta(\omega)$$

$$\begin{aligned}
C_{\Delta_{13}\Delta_{13}}[\omega] &= C_{\Delta_{23}\Delta_{23}}[\omega] = C_{\Delta_{12}\Delta_{12}}[\omega] \\
&= C_{\Delta'_{12}\Delta'_{12}}[\omega] = \frac{3}{4}C_{\Delta_{33}\Delta_{33}}[\omega]
\end{aligned}$$

When compared to foregoing expressions, higher order curvature fluctuations due to gravity of vacuum correspond to small modifications scaling as $\omega^5 l_P^4$. A new feature associated with these higher order terms is that fluctuations of the trace component ($\Delta = \Delta_\mu^\mu$) no longer vanish

$$C_{\Delta\Delta}[\omega] = \frac{8(4 + N_\nu)}{105\pi}\omega^5 l_P^4 \sin^2\frac{\omega\tau}{2}\theta(\omega)$$

Notice that this noise spectrum would not be affected by the contribution of gravitational stress tensor, since Δ is obtained as a contraction of Ricci tensor (see the discussion in the end of section 5). It is worth emphasizing that this expression depends on the number N_ν of massless neutrino fields, precisely of neutrino fields with a mass smaller than the analysis frequency.

In the context of this paper where attention is restricted to fluctuations and dissipation at experimentally accessible frequencies, this last result establishes a direct connection between fundamental matter fields and minimal fluctuations of spacetime. This connection has been derived here from a few minimal properties of gravitational quantum fluctuations, namely the relations between response functions and fluctuations and the known effective behaviour of gravitation at low frequencies. It should therefore subsist in a consistent theory including quantum gravity.

APPENDIX A: STRESS TENSOR CORRELATION FUNCTIONS FOR A SPACETIME OF ARBITRARY DIMENSION

Stress tensor correlation functions can be written for a spacetime of arbitrary integer dimension d ($d \geq 2$). A purely dimensional effect appears in expression (10) of σ. Coefficients ζ_r also depend upon the dimension and the specific quantum field theory studied.

Massless scalar fields correspond to

$$\zeta_0 = (4\pi)^{-d/2}\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d+2)}$$

$$\zeta_1 = \frac{(d-2)^2(d+1)}{2}\zeta_0$$

The particular case of a two-dimensional spacetime requires a special attention. Explicit computation gives

$$\sigma_{T_{\mu\nu}T_{\rho\sigma}} = \frac{\hbar}{24}\pi^0_{\mu\nu\rho\sigma}k^2\theta(k^2) + \frac{\hbar}{12}k_\mu k_\nu k_\rho k_\sigma \delta(k^2)$$

This expression obeys condition (9) associated with energy conservation. In contrast with the generic case however, an extra contribution describing fluctuations

with light-like wavevectors appears superimposed to the π^r-terms ($\zeta_1 = 0$ and $\zeta_0 = \frac{1}{24\pi}$ in this case). This reveals the anomalous character of results specific to a two-dimensional spacetime.

It is possible to compute the functions ζ_r for massive fields (mass μ; $\zeta_r\{\mu=0\}$ is given above)

$$\zeta_r = \zeta_r\{\mu=0\}\theta(k^2-4\mu^2) \\ \times \left(1-4\frac{\mu^2}{k^2}\right)^{(d-3)/2}\left(1-4\lambda_r\frac{\mu^2}{k^2}\right)^2$$

Note that positive values are obtained for any values of momenta.

One may also give stress tensor correlation functions for Maxwell fields

$$\zeta_0 = (4\pi)^{-d/2}\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d+2)}(2d^2-3d-8)$$
$$\zeta_1 = \frac{(4\pi)^{-d/2}}{2}\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d+2)}(d-4)^2(d-2)(d+1)$$

For two-dimensional spacetime, ζ_0 is negative, in contrast with the generic positiveness property of coefficients ζ_r. In this case however, Weyl curvatures vanish and do not appear in stress tensor response.

We get in particular for massless scalar fields in four-dimensional spacetime (same result as in [28])

$$\zeta_0 = \frac{1}{960\pi^2} \qquad \zeta_1 = 10\zeta_0$$

and for Maxwell fields in four-dimensional spacetime (same result as in [27,28])

$$\zeta_0 = \frac{1}{80\pi^2} \qquad \zeta_1 = 0$$

Coefficient ζ_1 vanishes in this latter case, as a consequence of conformal invariance of Maxwell equations. It also vanishes for massless scalar fields in a two-dimensional spacetime, for the same reason.

APPENDIX B: EFFECTIVE GRAVITATION AND VACUUM STRESS TENSOR POLARIZATION

Effective gravitation theory may be modified by vacuum stress tensor polarization, if $\gamma_1(0)$ and g0(0) differ from the standard value 1. Some results of the paper have to be changed in this case. The graviton propagator is obtained as

$$\chi_{h_{\mu\nu}h_{\rho\sigma}} = \frac{2\kappa}{k^2-i\varepsilon}\sum(\lambda_r\gamma_r\pi^r_{\mu\nu\rho\sigma}) + \ldots$$

and corresponds to a modified gravitation equation

$$G^r_{\mu\nu} = \kappa\gamma_r T^r_{\mu\nu}$$

Defining an effective gravitational constant

$$\kappa_{\text{eff}} = \kappa\gamma_0(0)$$

and a parameter measuring deviation from general relativity

$$\delta\gamma_1 = \frac{\gamma_1(0)-\gamma_0(0)}{\gamma_0(0)}$$

we rewrite a modified Einstein equation

$$G^r_{\mu\nu} = \kappa_{\text{eff}}(T_{\mu\nu} + \delta\gamma_1 T^1_{\mu\nu})$$

As $T^1_{\mu\nu}$ is proportional to the traced stress tensor, this equation looks like a scalar-tensor equation of gravitation [32].

For evaluating the deviation from general relativity, we study the metric perturbation in the field of a static point-like mass m

$$h_{00}[k] = 2\pi\delta(k_0)\frac{\kappa_{\text{eff}}m}{k^2}\frac{2}{d-2}\left(d-3-\frac{\delta\gamma_1}{d-1}\right)$$
$$h_{ii}[k] = 2\pi\delta(k_0)\frac{\kappa_{\text{eff}}m}{k^2}\frac{2}{d-2}\left(1+\frac{\delta\gamma_1}{d-1}\right)$$

Latin indices represent spatial components and metric tensor is evaluated at first order in m with a gauge choice such that it is diagonal. We then compute Eddington parameter γ as the ratio between the spatial and temporal perturbations, and obtain for $d=4$

$$\gamma = \frac{1+\frac{\delta\gamma_1}{3}}{1-\frac{\delta\gamma_1}{3}}$$

Experiments on light bending and gravitational time delays [32] tell us that $\delta\gamma_1$ has a small value and then that $\gamma_1(0)$ and $\gamma_0(0)$ are close to each other.

Correlation functions for proper fluctuations of Riemann curvatures are given by equations (7) with modified coefficients a and b

$$a = 2\pi\kappa\delta(k^2)\sum(\lambda_r\gamma_r\alpha_r) \\ = -\frac{2}{d-2}\pi\kappa_{\text{eff}}\delta(k^2)\left(1+\frac{\delta\gamma_1}{d-1}\right)$$
$$b = 2\pi\kappa\delta(k^2)\sum(\lambda_r\gamma_r\beta_r) \\ = \pi\kappa_{\text{eff}}\delta(k^2)$$

Einstein tensor does no longer vanish but the curvature tensor $\sum\frac{G^r_{\mu\nu}}{\gamma_r}$ is directly proportional to the stress tensor and therefore has vanishing fluctuations.

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